# On the Orthogonal Basis of the Symmetry Classes of Tensors Associated with Certain Characters

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**Abstract**

A necessary condition is given for the existence of an O-basis for the symmetry classes of tensors associated with a finite group and the irreducible constituents of the permutation character of the finite group. This extends a result of R. R. Holmes [see the Main theorem of Linear and Multilinear Algebra 39 (1995), 241-243].

**Keywords:** Symmetry class of tensors, Orthogonal basis, Permutation character, Irreducible constituent.

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## 1 Introduction

Let $V$ be an $m$-dimensional vector space over the complex field $\mathbb{C}$. Let $\overset{n}{\otimes}V$ be the $n$th tensor power of $V$ and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation $\sigma$ in $\$_n$ there corresponds a unique linear operator $P(\sigma)$: $\overset{n}{\otimes}V \to \overset{n}{\otimes}V$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Let $G$ be a subgroup of $\$_n$ and let $\mathrm{Irr}(G)$ be the set of all the irreducible complex characters of $G$. It follows from the orthogonality relations for characters that

$$\left\{T(G,\chi):\ \overset{n}{\otimes}V \to \overset{n}{\otimes}V \middle|\ T(G,\chi) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)P(\sigma),\ \chi\in\mathrm{Irr}(G)\right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes}V$ under the $T(G,\chi)$ is called the *symmetry class of tensors* associated with $G$ and $\chi$ and is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \ldots * v_n$ and is called a *decomposable symmetrized tensor*.

Let $\Gamma_m^n$ be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \le \alpha_i \le m$. Then the group $G$ acts on $\Gamma_m^n$ by $\sigma \cdot \alpha = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$, where $\sigma \in G$ and $\alpha \in \Gamma_m^n$. Let $O(\alpha) = \{\sigma\cdot\alpha |\ \sigma \in G\}$ be the *orbit* of $\alpha$, and $G_\alpha$ be its *stabilizer subgroup*, i.e., $G_\alpha = \{\sigma \in$

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$G|\ \sigma \cdot \alpha = \alpha\}$, and consider a system $\Delta$ of distinct representatives of the orbits of $\Gamma_m^n$.

Let $\{e_1, \ldots, e_m\}$ be a basis of $V$. Denote by $e_\alpha^*$ the tensor $e_{\alpha_1} * \ldots * e_{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_m^n$. For $\alpha \in \Delta$, $V_\alpha^* = \langle e_{\sigma\cdot\alpha}^* |\ \sigma \in G\rangle$ is called the *orbital subspace* of $V_\chi^n(G)$, and we can easily prove that

$$V_\chi^n(G) = \bigoplus_{\alpha\in\Delta} V_\alpha^*. \tag{1}$$

Note that it is possible for some $\alpha \in \Delta$ to have $V_\alpha^* = 0$. But Freese (see [2]) proved that for $\alpha \in \Delta$

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma\in G_\alpha} \chi(\sigma), \tag{2}$$

therefore, if we consider

$$\overline{\Delta} = \Big\{\alpha \in \Delta |\ \sum_{\sigma\in G_\alpha} \chi(\sigma) \neq 0\Big\},$$

then the sum in (1) will be reduced to the form

$$V_\chi^n(G) = \bigoplus_{\alpha\in\overline{\Delta}} V_\alpha^*. \tag{3}$$

Of course we define the right-hand side of (3) to be 0, if $\overline{\Delta} = \emptyset$.

A particular case appears when we assume that $V$ is an $m$-unitary space. In this case, the inner product on $V$ induces an inner product on $\overset{n}{\otimes} V$, whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \ldots * u_n | v_1 * \ldots * v_n\rangle = \frac{\chi(1)}{|G|} \sum_{\sigma\in G} \chi(\sigma) \prod_{i=1}^n A_{i\sigma(i)},$$

where $A = [A_{ij}]_{n\times n} = [\langle u_i | v_j\rangle]_{n\times n}$.

If $\{e_1, \ldots, e_m\}$ is an orthonormal basis of $V$, then the sum appeared in (3) is an orthogonal direct sum. In addition, we can obtain

$$\langle e_{g\cdot\alpha}^* | e_{g'\cdot\alpha}^*\rangle = \frac{\chi(1)}{|G|} \sum_{\sigma\in G_\alpha} \chi(g'\sigma g^{-1}). \tag{4}$$

Let $W$ be a subspace of $V_\chi^n(G)$. An orthogonal basis of $W$ of the form $\{e_\alpha^* |\ \alpha \in S\}$, where $S$ is a subset of $\Gamma_m^n$, is called an *O-basis* of $W$. Because the sum appeared in (3) is an orthogonal direct sum, $V_\chi^n(G)$ has an O-basis if and only if for all $\alpha \in \overline{\Delta}$, the orbital subspace $V_\alpha^*$ has an O-basis. Note that, if $\chi$ is of degree one, since $\dim V_\alpha^* = 1$ for all

$\alpha \in \overline{\Delta}$, then $V_\alpha^*$ has an O-basis for all $\alpha \in \overline{\Delta}$, which implies that $V_\chi^n(G)$ has such a basis.

Several papers are devoted to the investigation of the existence of an O-basis for $V_\chi^n(G)$, see for example [6]. In [4] a necessary and sufficient condition for the existing of an O-basis for $V_\chi^n(G)$ is given, where $G$ is a cyclic or a dihedral group and in [1] when $G$ is a dicyclic group.

The main purpose of this article is to generalize a result of Holmes (see [3, Main theorem]). In fact, he proved that if $G$ is a 2-transitive subgroup of $\$_n$, $n \geq 3$, $\chi = \theta - 1_G$, where $\theta$ is permutation character of $G$, and $m \geq 2$, then $V_\chi^n(G)$ does not have an O-basis.

In the following section, we will omit condition of "2-transitivity of $G$" and will find a necessary condition for the existence of an O-basis for the symmetry class of tensors associated with $G$ and any constituent of its permutation character. This generalization will be independent of the permutation structure of the group $G$.

## 2 Results

Let $V$ be an $m$-unitary space, $G$ a finite group, and $\Omega$ a set of n elements. Suppose $G$ acts faithfully on $\Omega$, so we can assume that $G$ is a subgroup of $\$_n$. In fact, we consider $\{f_\sigma |\ \sigma \in G\}$ as the group $G$, where $f_\sigma : \Omega \longrightarrow \Omega$ defined by $f_\sigma(\omega) = \sigma \cdot \omega$ for all $\omega \in \Omega$, is a permutation on $n$ letters. Therefore the inner product space $V_\chi^n(G)$ is meaningful for all $\chi \in \mathrm{Irr}(G)$. Denote the permutation character of $G$ by $\theta$. For $\sigma \in G$, the value $\theta(\sigma)$ is the number of letters fixed by $\sigma$, i.e., $\theta(\sigma) = |\{\omega \in \Omega |\ \sigma \cdot \omega = \omega\}|$.

**Main Theorem** *Let $G$ be a finite group and let $\Omega$ be a set of $n$ elements, $n \geq 2$. Assume that $G$ acts transitively and faithfully on $\Omega$ and let $V$ be an $m$-unitary space, $m \geq 2$. Let $\chi$ be an irreducible constituent of $\theta$, where $\theta$ is the permutation character of $G$. If $\chi(1)(\chi, \theta)_G > \frac{1}{2}n$, then $V_\chi^n(G)$ does not have an O-basis.*

*Proof.* Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of $V$. Suppose $V_\chi^n(G)$ has an O-basis. Then, by (3), for all $\alpha \in \overline{\Delta}$, the orbital subspace $V_\alpha^*$ has an O-basis. Put $\alpha = (1, 2, \dots, 2)$, then $n \geq 2$ and $m \geq 2$ implies that $\alpha \in \Gamma_m^n$. Consider the action of $G$ on $\Gamma_m^n$ and choose $\Delta$ such that $\alpha \in \Delta$.

Without loss of generality, we can assume that $\Omega = \{1, 2, \dots, n\}$ and therefore $G_\alpha = G_1$, where $G_\alpha$ is the stabilizer subgroup of $\alpha$ when $G$ acts on $\Gamma_m^n$ and $G_1$ is the stabilizer subgroup of 1 when $G$ acts on $\Omega$. So by Lemma 5.14 of [5] we obtain $1_{G_\alpha} \uparrow^G = 1_{G_1} \uparrow^G = \theta$.

Frobenius reciprocity implies that

$$\sum_{\sigma \in G_\alpha} \chi(\sigma) = |G_\alpha|(\chi \downarrow_{G_\alpha}, 1_{G_\alpha})_{G_\alpha} = |G_\alpha|(\chi, 1_{G_\alpha} \uparrow^G)_G = |G_\alpha|(\chi, \theta)_G, \tag{5}$$

so by hypotheses, $(\chi, \theta)_G \neq 0$, and we obtain $\sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0$ and $\alpha \in \overline{\Delta}$. By the above discussion, $V_\alpha^*$, $\alpha = (1, 2, \ldots, 2)$, has an O-basis. We now have $[G : G_\alpha] = [G : G_1] = |\Omega| = n$, so $G = \bigcup_{i=1}^n g_i G_\alpha$, where $\{g_1, \ldots, g_n\}$ is a system of distinct representatives of left cosets of $G_\alpha$ in $G$.

Since, by (2) and (5),

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \chi(1)(\chi, \theta)_G =: s,$$

we can assume that $\{e^*_{g_1 \cdot \alpha}, e^*_{g_2 \cdot \alpha}, \ldots, e^*_{g_s \cdot \alpha}\}$ is an O-basis for $V_\alpha^*$.

Define the $n \times n$ complex matrix $A = [A_{ij}]$ by

$$A_{ij} = \langle e^*_{g_i \cdot \alpha} | e^*_{g_j \cdot \alpha} \rangle.$$

Note that $s < n$, therefore $A$ may be written in the form

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where $A_1$, $A_2$, $A_3$, and $A_4$ are matrices of sizes $s \times s$, $s \times (n-s)$, $(n-s) \times s$, and $(n-s) \times (n-s)$, respectively. But, by (4), for $i, j, 1 \leq i, j \leq s$, we have

$$\begin{aligned} A_{ij} &= \langle e^*_{g_i \cdot \alpha} | e^*_{g_j \cdot \alpha} \rangle \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\chi(1)}{|G|} \sum\limits_{\sigma \in G_\alpha} \chi(\sigma) & \text{if } i = j, \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{[G:G_\alpha]} \dim V_\alpha^* & \text{if } i = j, \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{s}{n} & \text{if } i = j, \end{cases} \\ &= (\tfrac{s}{n} I_s)_{ij}, \end{aligned}$$

where $I_s$ is the $s \times s$ identity matrix. So the block form of $A$ has been changed to the form

$$A = \begin{bmatrix} \frac{s}{n} I_s & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

We now claim that $A$ is idempotent. In fact,

$$\begin{aligned}
(A^2)_{ij} &= \sum_{\ell=1}^{n} A_{i\ell} A_{\ell j} \\
&= \sum_{\ell=1}^{n} \langle e^*_{g_i \cdot \alpha} | e^*_{g_\ell \cdot \alpha} \rangle \langle e^*_{g_\ell \cdot \alpha} | e^*_{g_j \cdot \alpha} \rangle \\
&= \sum_{\ell=1}^{n} \left( \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(g_\ell \sigma g_i^{-1}) \right) \left( \frac{\chi(1)}{|G|} \sum_{\tau \in G_\alpha} \chi(g_j \tau g_\ell^{-1}) \right) \\
&= \tfrac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^{n} \sum_{\sigma \in G_\alpha} \sum_{\tau \in G_\alpha} \chi(g_\ell \sigma g_i^{-1}) \chi(g_j \tau g_\ell^{-1}) \\
&= \tfrac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^{n} \sum_{\lambda \in g_\ell G_\alpha} \sum_{\mu \in G_\alpha g_\ell^{-1}} \chi(\lambda g_i^{-1}) \chi(g_j \mu) \\
&= \tfrac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^{n} \sum_{\lambda \in g_\ell G_\alpha} \sum_{\sigma \in G_\alpha} \chi(\lambda g_i^{-1}) \chi(g_j \sigma \lambda^{-1}) \\
&= \tfrac{\chi(1)^2}{|G|^2} \sum_{\lambda \in G} \sum_{\sigma \in G_\alpha} \chi(\lambda g_i^{-1}) \chi(g_j \sigma \lambda^{-1}) \\
&= \tfrac{\chi(1)^2}{|G|^2} \sum_{\lambda \in G} \sum_{\sigma \in G_\alpha} \chi(\lambda) \chi(g_j \sigma g_i^{-1} \lambda^{-1}) \\
&= \tfrac{\chi(1)^2}{|G|^2} \sum_{\sigma \in G_\alpha} \sum_{\lambda \in G} \chi(\lambda) \chi(g_j \sigma g_i^{-1} \lambda^{-1}) \\
&= \tfrac{\chi(1)^2}{|G|^2} \sum_{\sigma \in G_\alpha} \left( \frac{|G|}{\chi(1)} \chi(g_j \sigma g_i^{-1}) \right) \\
&= \tfrac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(g_j \sigma g_i^{-1})
\end{aligned}$$

$$= \langle e^*_{g_i\cdot\alpha} | e^*_{g_j\cdot\alpha}\rangle$$

$$= A_{ij}.$$

Therefore, the claim holds. Now, using $A^2 = A$, we obtain

$$A_2A_3 = (\frac{s}{n} - \frac{s^2}{n^2})I_s.$$

Since $0 < s < n$, $A_2A_3$ is an invertible matrix and hence $s \leq n-s$, $s \leq \frac{1}{2}n$, or $\chi(1)(\chi,\theta)_G \leq \frac{1}{2}n$, which is a contradiction. Thus $V^n_\chi(G)$ does not have an O-basis. $\square$

**Corollary 1** (Main theorem [3]) *Let $G$ be a 2-transitive subgroup of $\$_n$, $n \geq 3$. Let $\chi = \theta - 1_G$, where $\theta$ is the permutation character of $G$. If $dimV = m \geq 2$, then $V^n_\chi(G)$ does not have an O-basis.*

*Proof.* $G$ has a natural transitive and faithful action on the set $\Omega = \{1, 2, \ldots, n\}$, given by $\sigma \cdot\ i = \sigma(i)$. 2-Transitivity of this action implies that $\chi$ is an irreducible constituent of $\theta$ . On the other hand, $n \geq 3$ and so $\chi(1)(\chi,\theta)_G = (n-1) > \frac{1}{2}n$. Hence, by our main theorem, $V^n_\chi(G)$ does not have an O-basis. $\square$

**Example** Let $G$=$\Omega$=$\mathbb{A}_4$ be the alternating group of degree 4. We know that $G$ acts transitively and faithfully on $\Omega$ by right multiplication, so $G$ is a subgroup of $\$_{12}$. $G$ has an irreducible complex character, say $\chi$, of degree 3. As $\theta$, the permutation character of $G$, is regular, $\chi$ is an irreducible constituent of $\theta$ of multiplicity 3. Therefore, $\chi(1)(\chi,\theta)_G = 9 > 6 = \frac{1}{2}|\Omega|$, so by the main theorem $V^{12}_\chi(G)$ does not have an O-basis. Note that in this example the action of $G$ on $\Omega$ is not 2-transitive, so our main result really extends the result of [3].

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